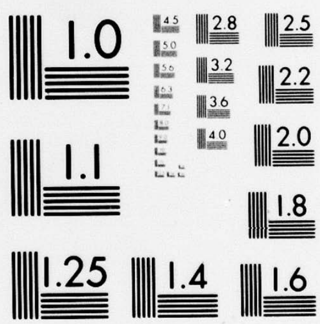
AD-A073 814 ILLINOIS UNIV AT CHICAGO CIRCLE DEPT OF MATHEMATICS F/G 12/1
EXTENSIONS OF A SIMPLE MODEL WITH APPLICATIONS IN RELIABILITY, --ETC(U)
JUL 79 E EL-NEWEIHI AFOSR-76-3050
UNCLASSIFIED AFOSR-TR-79-0975 NL

| OF |

AD
A073814

END
DATE
FILMED
10-79
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFOSR-TR- 79-0975

AD A073814

DDC FILE COPY

D D C
SEP 17 1979
C

Approved for public release;
distribution unlimited.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 79-0975 (18) (19)	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER (9)	
4. TITLE (and Subtitle) (6): EXTENSIONS OF A SIMPLE MODEL WITH APPLICATIONS IN RELIABILITY, EXTINCTION OF SPECIES, INVENTORY DEPLETION AND URN SAMPLING.	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10): Emad El-Neweihi	8. CONTRACT OR GRANT NUMBER(s) (15): AFOSR-76-3050
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Illinois at Chicago Circle, Department of Mathematics, Box 4348, Chicago, IL 60680	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F (16) 2304 (17) A5
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE (11): July 1979
	13. NUMBER OF PAGES: 17
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

LEVEL IV

2

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(12) 21p.

DDC
RECEIVED
SEP 17 1979
C

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

reliability; extinction of species; inventory; sampling, Schur functions, cut sets, series-parallel system, functions decreasing in transposition.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is devoted to the study of the following model: A series-parallel system consists of $(k+1)$ subsystem $C_0, C_1, \ldots, C_k$, also called cut sets. Cut set C_i has n_i components arranged in parallel, $i = 0,1,\ldots,k$. No cut sets have a component in common. This model was introduced and studied by El-Neweihi, Proschan, and Sethuraman (1978) under the assumption

next page (CONTINUED)

DD FORM 1 JAN 73 1473

409 950

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DDC FILE COPY

AD A073814

20. Abstract continued.

that component lifelengths are continuous, independeny, and identically distributed random variables. They obtained several equivalent expressions for the probability that a specified cut set C_0, say, fails first. These expressions were then used to derive qualitative properties of this probability, such as monotonicity, Schur-concavity, etc.

In this paper ~~we~~ obtains extensions of these results. Under the same assumptions we study the probability that a specified cut set C_0^r, say, fails, in the r^{th} place, $r = 1,2,\ldots,k$. This probability is shown to retain most of the interesting qualitative features enjoyed in the special case $r = 1$. It is ~~We~~ then assume that component lifelengths are identically distributed within a cut set, but allowed ~~them~~ to vary among cut sets. Under this more general assumption ~~we~~ ~~derive~~ expressions are derived for and obtain properties of the probability that C_0^r fails in the r^{th} place.

This generalization of the model of El-Neweihi, Proschan, and Sethuraman (1978), also has applications in the study of reliability, extinction of species, inventory depletion, urn sampling, among others.

Accession For	
NTIS GRA&I	☑
DDC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or special
A	

EXTENSIONS OF A SIMPLE MODEL WITH APPLICATIONS IN RELIABILITY, EXTINCTION OF SPECIES, INVENTORY DEPLETION AND URN SAMPLING

by

Emad El-Neweihi*
Department of Mathematics
University of Illinois at Chicago Circle

July 1979

*Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, and Grant No. AFOSR 76-3050C.

<u>Key words and phrases</u>: reliability; extinction of species; inventory, sampling; Schur functions; cut sets, series-parallel system; functions decreasing in transposition.

ABSTRACT

This paper is devoted to the study of the following model: A series-parallel system consists of $(k + 1)$ subsystems C_0, C_1, ..., C_k, also called cut sets. Cut set C_i has n_i components arranged in parallel, $i = 0, 1, \ldots, k$. No two cut sets have a component in common. This model was introduced and studied by El-Neweihi, Proschan, and Sethuraman (1978) under the assumption that component lifelengths are continuous, independent, and identically distributed random variables. They obtained several equivalent expressions for the probability that a specified cut set C_0, say, fails first. These expressions were then used to derive qualitative properties of this probability, such as monotonicity, Schur-concavity, etc.

In this paper we obtain extensions of these results. Under the same assumptions we study the probability that a specified cut set C_0, say, fails in the $r^{\underline{th}}$ place, $r = 1, 2, \ldots, k$. This probability is shown to retian most of the interesting qualitative features enjoyed in the special case $r = 1$. We then assume that component lifelengths are identically distributed within a cut set, but allow them to vary among cut sets. Under this more general assumption we derive expressions for and obtain properties of the probability that C_0 fails in the $r^{\underline{th}}$ place.

This generalization of the model of El-Neweihi, Proschan, and Sethuraman (1978), also has applications in the study of reliability, extinction of species, inventory depletion, urn sampling, among others.

1. INTRODUCTION AND SUMMARY.

This paper is devoted to the study of a simple model which has applications in reliability theory, extinction of species, inventory depletion, and urn sampling. The model was introduced and studied by El-Neweihi, Proschan, and Sethuraman (1978), (referred to as EPS throughout). In this paper various extensions of their results are obtained. We state the model in a reliability context and use the language of reliability theory in the derivation of the results.

Consider a system consisting of $k + 1$ subsystems, called cut sets, arranged in series. The $i^{\underline{th}}$ cut set, called C_i, has n_i components arranged in parallel, $i = 0, 1, \ldots, k$. No two cut sets have a component in common. Such a system is called a series-parallel system. (A series system functions if and only if each component in it functions. A parallel system functions if and only if at least one of its components functions.) When little is known about the life distribution of the components of a series-parallel system, it is reasonable to assume that after t components have failed, each of the remaining components are equally likely to fail. It is also further assumed that components fail one at a time. These assumptions are satisfied by considering the lifelengths of the components to be exchangeable random variables, in particular, independent and identically distributed random variables. Under these assumptions, EPS study $P(n_0; \underline{n})$, the probability that cut set C_0 fails first (causing the failure of the system), where $\underline{n} = (n_1, \ldots, n_k)$ represent the vector of sizes of cut sets $C_1, \ldots, C_k$. Several alternative expressions for $P(n_0; \underline{n})$ are obtained and interesting qualitative features are derived.

In the present paper we study $P^r(n_0; \underline{n})$, the probability that cut set C_0 fails in the $r^{\underline{th}}$ place, $r = 1, 2, \ldots, k$, both under the same assumptions described above and under verious relaxations of these assumptions as described below. Note that $P(n_0, \underline{n})$ is now the special case corresponding to $r = 1$.

The organization of this paper is as follows. Section 2 contains the notation used throughout the paper. Section 3 contains expressions for $P^r(n_0; \underline{n})$ and various interesting qualitative features. In this section the component lifelengths are assumed to be continuous independent and identically distributed random variables. In Section 4 we still assume the component lifelengths to be continuous and independent, however we now require them to be identically distributed only within a cut set, but allow them to vary among cut sets. Under this relaxation of the assumption, we obtain expressions for $P^r(n_0; \underline{n})$ and derive some of the interesting qualitative features that $P^1(n_0; \underline{n})$ possesses.

We now describe briefly but explicitly how the results of this paper have applications in other practical areas.

a) Inventory depletion. A depot stocks $(k + 1)$ brands of a certain item. Under appropriate interpretation of the assumptions, the probability that brand 0 is the $r^{\underline{th}}$ to be depleted is $P^r(n_0; \underline{n})$.

b) Sampling from urns. An urn contains n_i balls of color i, $i = 0, 1, \ldots, k$. Balls are removed at random one by one. The probability that color 0 is the $r^{\underline{th}}$ color to be exhausted is again $P^r(n_0; \underline{n})$.

It is clear that the model occurs in other contexts as well.

2. NOTATION

We use the following notation throughout. n denotes the total number of components in the series-parallel system. S denotes a subset of $\{1, \ldots, k\}$, possibly the empty subset.

$|S|$ = the cardinality of S.

$S' = \{i: \ 1 \leq i \leq k, \ i \notin S\}$.

$n_S = \sum_{i \in S} n_i$ and $n_S = 0$ for $S = \phi$.

$\underline{n}_i = (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k)$.

$\binom{a}{b} = 0$ when $b < 0$.

3. EXPRESSIONS FOR AND QUALITATIVE PROPERTIES OF $P^r(n_0; \underline{n})$.

In this section we study $P^r(n_0; \underline{n})$, $r = 1, \ldots, k$, under the assumption that the lifelengths of the n components are independent and identically distributed with a common continuous distribution. Then the n! patterns of component failures are equally likely. Under the same assumption, EPS derive various expressions and recurrence relations for $P^1(n_0; \underline{n})$ using different techniques, such as conditioning, augmentations, and order statistic theory. In a similar fashion, we derive for $P^r(n_0; \underline{n})$ several expressions and recurrence relations. We give only a sample of such results in Theorem 3.1 and Theorem 3.2 below. We do concentrate however on the expression derived in Theorem 3.3 using order statistics techniques since it is compact and mathematically more tractable than the alternative expressions. Note that the case $r = k + 1$ is trivial ($P^{k+1}(n_0, \underline{n}) = \frac{n_o}{n}$) and is therefore not discussed.

For any series-parallel system with cut sets $D_1, \ldots, D_\ell$, let $\{D_1 < D_2 < \ldots < D_\ell\}$ denote the event that cut set D_1 fails first, D_2 fails second, ..., and D_ℓ fails $\ell^{\underline{th}}$.

<u>Theorem 3.1</u>. Let $1 \le r \le k$. Then

$$P^r(n_0; \underline{n}) = \sum_\pi \frac{n_{\pi_r} \cdot n_{\pi_{r+1}} \cdots n_{\pi_k}}{(n_0+\ldots+n_{\pi_r})\ldots(n_0+\ldots+n_{\pi_r}+\ldots+n_{\pi_k})} \cdot \frac{n_0\, n_{\pi_2} \cdots n_{\pi_{r-1}}}{(n_{\pi_1}+n_{\pi_2})\ldots(n_{\pi_1}+\ldots+n_{\pi_{r-1}})(n_{\pi_1}+\ldots+n_{\pi_{r-1}}+n_0)}, \tag{3.1}$$

where the summation is over all permutations $\pi = (\pi_1, \ldots, \pi_k)$ of $(1, 2, \ldots, k)$.

<u>Proof</u>. By Theorem 3.2 of EPS,

$$P(C_{\pi_1} < \ldots < C_{\pi_{r-1}} < C_0 < C_{\pi_r} < \ldots < C_{\pi_k}) =$$

$$\frac{n_{\pi_r} \cdot n_{\pi_{r+1}} \cdots n_{\pi_k}}{(n_0+\ldots+n_{\pi_r})\ldots(n_0+\ldots+n_{\pi_r}+\ldots+n_{\pi_k})} \cdot$$

$$\frac{n_0\, n_{\pi_2} \cdots n_{\pi_{r-1}}}{(n_{\pi_1}+n_{\pi_2})\ldots(n_{\pi_1}+\ldots+n_{\pi_{r-1}})(n_{\pi_1}+\ldots+n_{\pi_{r-1}}+n_0)}.$$

The expression for $P^r(n_0; n)$ in (3.1) is obtained by summing the probabilities of the disjoint events $\{C_{\pi_1} < \ldots < C_{\pi_{r-1}} < C_0 < C_{\pi_r} < \ldots < C_{\pi_k}\}$ over all permutations π. Obvious adjustment in the expression in (3.1) for the case $r = 1$, $r = 2$ has to be made.

The next theorem illustrates the use of a conditioning argument to obtain recurrence relations for $P^r(n_0, \underline{n})$.

<u>Theorem 3.2</u>. Let $1 \le r \le k$. Then

$$(3.2)\quad P^r(n_0; \underline{n}) = \sum_{i=0}^{k} \frac{n_i}{n} P^r(n_0; (n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k))$$

and

$$(3.3)\quad P^r(n_0; \underline{n}) = \sum_{i=1}^{k} \frac{n_i}{n} P^r(n_0; \underline{n}_i).$$

<u>Proof</u>. Relation (3.2) is obtained by conditioning on the outcome at the first component failure. Thus,

$$P^r(n_0; \underline{n}) = P(C_0 \text{ fails } r^{\underline{th}}) = \sum_{i=0}^{k} P(C_0 \text{ fails } r^{\underline{th}} | \text{first component to fail comes}$$

from C_i). P(first component to fail comes from C_i) =

$\sum_{i=0}^{k} \frac{n_i}{n} P^r(n_{0j}(n_1, \ldots, n_{i-1}, n_i-1, n_{i+1}, \ldots, n_k))$. Relation (3.3) is similarly obtained by conditioning on the outcome of the last component failure.

We now obtain a compact and mathematically useful expression for $P^r(n_0; \underline{n})$ using an order statistics approach.

<u>Theorem 3.3</u>. Let $1 \le r \le k$. Then

$$(3.4)\quad P^r(n_0; \underline{n}) = \sum_{S:|S|=r-1} \int_0^1 x^{n_S} \prod_{i \in S'} (1 - x^{n_i}) n_0 x^{n_0 - 1}\, dx.$$

<u>Proof</u>. Let X_{ij} be the random variable representing the lifelength of component j in cut set i, $j = 1, \ldots, n_i$, $i = 0, 1, \ldots, k$. The X_{ij}'s are independent and identically distributed with a common continuous distribution. Assume that this common distribution is uniform on (0, 1). Defining

$X_i^* = \max\limits_{1 \le j \le n_i} X_{ij}$, $i = 0, 1, \ldots, k$, we have $P^r(n_0; \underline{n}) = \sum\limits_{S:|S|=r-1} P(\max\limits_{i \in S} X_i^* < X_0^* < \min\limits_{j \in S'} X_j^*)$.

Now $P(\max\limits_{i \in S} X_i^* < X_0^* < \min\limits_{j \in S'} X_j^*) = \int_0^1 P(\max\limits_{i \in S} X_i^* < x) P(\min\limits_{j \in S'} X_j^* > x) n_0 x^{n_0 - 1} dx$

$$= \int_0^1 x^{n_S} \prod_{j \in S'} (1 - x^{n_j}) n_0 \, x^{n_0 - 1} dx.$$

Relation (3.4) readily follows.

In the rest of this section, relation (3.4) provides the most mathematically tractable formula permitting us to obtain interesting properties of $P^r(n_0; \underline{n})$.

The next lemma treats the case where all the cut sets $C_1, \ldots, C_k$ or all but one of them have equal numbers of components.

Lemma 3.4. For $r = 1, \ldots, k$,

$$(3.5) \qquad P^r(n_0; \underbrace{m, \ldots, m}_{k}) = \frac{n_0}{m} \binom{k}{r-1} B\left(\frac{n_0 + m(r-1)}{m}, k - r + 2\right)$$

and

$$(3.6) \qquad P^r(n_0; p, \underbrace{m, \ldots, m}_{k-1}) = \binom{k-1}{r-2} \frac{n_0}{m} B\left(\frac{p + n_0 + m(r-2)}{m}, k - r + 2\right) +$$

$$\binom{k-1}{r-1} \frac{n_0}{m} \left[B\left(\frac{n_0 + m(r-1)}{m}, k - r + 1\right) - B\left(\frac{p + n_0 + m(r-1)}{m}\right), k - r + 1\right)\right],$$

where $B(.,.)$ is the usual beta functions.

Proof. From Theorem 3.3,

$$P^r(n_0; \underbrace{m, \ldots, m}_{k}) = \binom{k}{r-1} \int_0^1 x^{m(r-1)} (1 - x)^{k-r+1} n_0 x^{n_0 - 1} dx =$$

$$\frac{n_0}{m} \binom{k}{r-1} \int_0^1 y^{\frac{n_0}{m} + r - 2} (1 - y)^{k-r+2} dy = \frac{n_0}{m} \binom{k}{k-1} B\left(\frac{n_0 + m(r-1)}{m}\right), k - r + 1),$$

by substituting $y = x^m$. This establishes (3.5).

Also

$$P^r(n_0;\ \underbrace{p,\ m,\ \ldots,\ m}_{k-1}) = \sum_{S:|S|=r-1,p\in S} \int_0^1 x^{n_S} \prod_{j\in S'} (1 - x^{n_j}) n_0 x^{n_0-1}\,dx +$$

$$\sum_{S:|S|=r-1,p\in S'} \int_0^1 x^{n_S} \prod_{j\in S'} (1 - x^{n_j}) n_0 x^{n_j-1}\,dx = \binom{k-1}{r-2} \int_0^1 x^{m(r-2)+p}(1 - x^m)^{k-r+1} n_0 x^{n_0-1}\,dx$$

$$+ \binom{k-1}{r-1} \int_0^1 x^{m(r-1)}(1 - x^m)^{k-r}(1 - x^p) n_0 x^{n_0-1}\,dx.$$

Again by substituting $x^m = y$, relation (3.6) is readily obtained.

Lemma 3.4 is useful in establishing bounds on $P^r(n_0;\ \underline{n})$, as will be illustrated later in this section.

We now make precise several intuitively obvious properties of $P^r(n_0;\ \underline{n})$. These properties can provide a theoretical basis for setting up suitable maintenance and inspection policies to guard against failure of "weaker" cut sets, that is those having higher probabilities of failing early. In the inventory context such properties provide a theoretical basis for reordering policies.

We first note that $P^r(n_0;\ \underline{n})$ is permutation invariant in $(n_1, \ldots, n_k)$. For the special case $r = 1$, $P^r(n_0;\ \underline{n})$ is strictly increasing in each of the arguments $n_1, \ldots, n_k$ for fixed n_0, and strictly decreasing in n_0 for fixed $n_1, \ldots, n_k$. For $2 \le r \le k$, no such general monotonicity property is valid, as can be easily illustrated by examples.

The next theorem establishes a homogeneity property of $P^r(n_0;\ \underline{n})$, which is, after some reflection, intuitively clear.

Theorem 3.5. Let m be a positive integer and let $m\underline{n} = (mn_1, \ldots, mn_k)$. Then

(3.7) $$P^r(n_0;\ \underline{n}) = P^r(mn_0;\ m\underline{n}),\ r = 1, \ldots, k.$$

Proof. From (3.4),

$$P^r(mn_0, \underline{mn}) = \sum_{S:|S|=r-1} \int_0^1 x^{mn_S} \prod_{i\in S'} (1 - x^{mn_i}) mn_0 x^{mn_0-1} dx$$

$$= \sum_{S:|S|=r-1} \int_0^1 y^{n_S} \prod_{i\in S'} (1 - y^{n_i}) n_0 y^{n_0-1} dy = P^r(n_0; \underline{n}),$$

wherein we have substituted $y = x^m$.

Theorem 3.5 allows us to extend the domain of the function $P^r(n_0; \underline{n})$ as follows:

$$P^r(\lambda_0; \underline{\lambda}) = \sum_{S:|S|=r-1} \int_0^1 x^{(\sum_{i\in S}\lambda_i)} \prod_{j\in S'} (1 - x^{\lambda_j}) \lambda_0 x^{\lambda_0-1} dx \text{ for } \lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_k \geq 0$$

$P^r(\lambda_0; \underline{\lambda})$ represents the probability that a cut set C_0 fails in the $r^{\underline{th}}$ place in a series-parallel system when all the λ_i's are rational numbers. For general λ_i's, $P(\lambda_0; \underline{\lambda})$ would represent a limit of such probabilities. This last conclusion follows in the same way that this result was established for $P^1(n_0, \underline{n})$ by EPS.

We now show that $P^r(n_0; \underline{n})$ has a further special ordering property called Schur-concavity. We first recall the concepts of majorization and Schur-functions. Majorization (see Definition 3.6) is a partial ordering on R_k, k-dimensional Euclidean space. A Schur function is a function that is monotone with respect to this partial ordering. Many well-known inequalities arising in probability and statistics are equivalent to saying that certain functions are Schur functions. We now give definitions of majorization and Schur functions.

Definition 3.6. Given a vector $\underline{x} = (x_1, \ldots, x_k)$, let $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[k]}$ denote a decreasing rearrangement of $x_1, \ldots, x_k$. A vector $\underline{x}$ is said to majorize a vector $\underline{x}'$ (in symbols, $\underline{x} \overset{m}{\geq} \underline{x}'$) if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} , \quad j = 1, 2, \ldots, R - 1,$$

and

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} x_i'.$$

Note that $\underline{x} \overset{m}{\geq} \underline{x}'$ and $\underline{x}' \overset{m}{\geq} \underline{x}$ if and only if $\underline{x}'$ is a permutation of $\underline{x}$.

A useful characterization of majorization is given by Hardy, Littlewood, and Pólya (1952), p. 47.

<u>Lemma 3.7</u>. $\underline{x} \overset{m}{\geq} \underline{x}'$ if and only if there exists a finite number, say t, of vectors $\underline{x}^{(1)}, \ldots, \underline{x}^{(t)}$ such that $\underline{x} = \underline{x}^{(1)} \overset{m}{\geq} \underline{x}^{(2)} \geq \ldots \geq \underline{x}^{(t)} = \underline{x}'$ and such that $\underline{x}^{(i)}, \underline{x}^{(i+1)}$ differ in two coordinates only $i = 1, \ldots, t - 1$.

<u>Definition 3.8</u>. A function $f\colon R_k \to R$ is said to be a <u>Schur-convex</u> <u>(Schur-concave)</u> function if $f(\underline{x}) \geq (\leq) f(\underline{x}')$ whenever $\underline{x} \overset{m}{\geq} \underline{x}'$. Functions which are either Schur-convex or Schur-concave are called <u>Schur-functions</u>. Note that a <u>Schur-function</u> f is necessarily permutation invariant.

The following theroem shows that $P^r(n_0; \underline{n})$ is a Schur-concave function in $\underline{n}$ for each n_0, $r = 1, \ldots, k$. This property is then utilized to develop bounds on $P^r(n_0; \underline{n})$.

<u>Theorem 3.8</u>. Let $1 \leq r \leq k$, and n_0 be fixed. Then $P^r(n_0, \underline{n})$ is Schur-concave in $\underline{n}$.

<u>Proof</u>. By Lemma 3.7 and the fact that $P^r(n_0; \underline{n})$ is permutation invariant, it suffices to show that $P^r(n_0; \underline{n}') \geq P^r(n_0; \underline{n})$, where $n_1 > n_2$, $n_1' = n_1 - 1$, $n_2' = n_2 + 1$, $n_j' = n_j$, $j = 3, \ldots, k$. Now

$$P^r(n_0; \underline{n}') - P^r(n_0; \underline{n}) = \sum_{S:|S|=r-1} \int_0^1 \Big[x^{n_S'} \prod_{j \in S'} (1 - x^{n_j'}) - x^{n_S} \prod_{j \in S'} (1 - x^{n_j})\Big] n_0 x^{n_0 - 1}\, dx.$$

We complete the proof by showing that the quantity between brackets inside the integral (call it h(x)) is non-negative. To see this consider the following three exclusive and exhaustive cases:

(i) $\{1, 2\} \subset S$; in this case h(x) is obviously equal to zero.

(ii) $\{1, 2\} \subset S'$: in this case $h(x) = x^{n'_S} \prod_{\substack{j \in S' \\ j \notin \{1,2\}}} (1 - x^{n'_j})[(1 - x^{n'_1})(1 - x^{n'_2}) - (1 - x^{n_1})(1 - x^{n_2})]$. Now $(1 - x^{n'_1})(1 - x^{n'_2}) - (1 - x^{n_1})(1 - x^{n_2}) = (x - 1)(x^{n_1 - 1} - x^{n_2}) \geq 0$, since $0 \leq x \leq 1$ and $n_1 - 1 \geq n_2$. This shows that $h(x) \geq 0$.

(iii) $1 \in S$ and $2 \in S'$ or $1 \in S'$ and $2 \in S$, we treat only one case since the other can be treated similarly. Now $x^{n'_1}(1 - x^{n'_2}) - x^{n_1}(1 - x^{n_2}) = x^{n_1 - 1} - x^{n_1} \geq 0$; it follows that $h(x) \geq 0$.

Thus $P^r(n_0; \underline{n}') \geq P^r(n_0; \underline{n})$; that is $P^r(n_0; \underline{n})$ is Schur-concave.

The above theorem can be explained intuitively as follows: when the total number of components is fixed, the more heterogeneous the sizes of cut sets $C_1, \ldots, C_k$, the more likely it is to find many "smaller" cut sets and many "larger" cut sets. The smaller cut sets make it harder for C_0 to fail in the r^{th} place for "small" r, and the larger cut sets make it harder for C_0 to fail in the $r^{\underline{th}}$ place for "large" r.

Note that since a limit of Schur-concave functions is Schur-concave, one can easily verify that $P^r(\lambda_0, \underline{\lambda})$ is Schur-concave in $\underline{\lambda}$ for fixed λ_0, $\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_k \geq 0$.

Finally, the next lemma gives bounds for $P^r(n_0; \underline{n})$ using the Schur-concave property and Lemma 3.4.

<u>Lemma 3.9</u>. Let $n_1, \ldots, n_k$ be positive numbers, $N = \sum_{i=1}^{k} n_i$, $n^* = \min(n_1, \ldots, n_k)$, $s = \frac{N}{k}$, and $p = N - (k - 1)n^*$. Then

We complete the proof by showing that the quantity between brackets inside the integral (call it h(x)) is non-negative. To see this consider the following three exclusive and exhaustive cases:

(i) $\{1, 2\} \subset S$; in this case h(x) is obviously equal to zero.

(ii) $\{1, 2\} \subset S'$: in this case $h(x) = x^{n'_S} \prod_{\substack{j \in S' \\ j \notin \{1,2\}}} (1 - x^{n'_j})[(1 - x^{n'_1})(1 - x^{n'_2}) - (1 - x^{n_1})(1 - x^{n_2})]$. Now $(1 - x^{n'_1})(1 - x^{n'_2}) - (1 - x^{n_1})(1 - x^{n_2}) = (x - 1)(x^{n_1 - 1} - x^{n_2}) \geq 0$, since $0 \leq x \leq 1$ and $n_1 - 1 \geq n_2$. This shows that $h(x) \geq 0$.

(iii) $1 \in S$ and $2 \in S'$ or $1 \in S'$ and $2 \in S$, we treat only one case since the other can be treated similarly. Now $x^{n'_1}(1 - x^{n'_2}) - x^{n_1}(1 - x^{n_2}) = x^{n_1 - 1} - x^{n_1} \geq 0$; it follows that $h(x) \geq 0$.

Thus $P^r(n_0; \underline{n}') \geq P^r(n_0; \underline{n})$; that is $P^r(n_0; \underline{n})$ is Schur-concave.

The above theorem can be explained intuitively as follows: when the total number of components is fixed, the more heterogeneous the sizes of cut sets $C_1, \ldots, C_k$, the more likely it is to find many "smaller" cut sets and many "larger" cut sets. The smaller cut sets make it harder for C_0 to fail in the r^{th} place for "small" r, and the larger cut sets make it harder for C_0 to fail in the $r^{\underline{th}}$ place for "large" r.

Note that since a limit of Schur-concave functions is Schur-concave, one can easily verify that $P^r(\lambda_0, \underline{\lambda})$ is Schur-concave in $\underline{\lambda}$ for fixed λ_0, $\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_k \geq 0$.

Finally, the next lemma gives bounds for $P^r(n_0; \underline{n})$ using the Schur-concave property and Lemma 3.4.

<u>Lemma 3.9</u>. Let $n_1, \ldots, n_k$ be positive numbers, $N = \sum_{i=1}^{k} n_i$, $n^* = \min(n_1, \ldots, n_k)$, $s = \frac{N}{k}$, and $p = N - (k - 1)n^*$. Then

(3.8) $$\binom{k-1}{r-2}\frac{n_0}{n^*}B\left(\frac{p+n_0+n^*(r-2)}{n^*}, k-r+2\right) +$$

$$\binom{k-1}{r-1}\frac{n_0}{n^*}\left[B\left(\frac{n_0+n^*(r-1)}{n^*}, k-r+1\right) - B\left(\frac{p+n_0+n^*(r-1)}{n^*}, k-r+1\right)\right]$$

$$\le P^r(n_0; \underline{n}) \le \frac{n_0}{s}\binom{k}{r-1}B\left(\frac{n_0+s(r-1)}{s}, k-r+2\right)$$

Proof. It is easy to see that $(p, \underbrace{n^*, \ldots, n^*}_{k-1}) \overset{m}{\ge} (n_1, \ldots, n_k) \overset{m}{\ge} (\underbrace{s, \ldots, s}_{k})$.

Relation (3.8) now follows from the Schur-concavity of $P^r(n_0; \underline{n})$ and Lemma 3.4.

4. FURTHER EXTENSIONS AND GENERALIZATIONS.

In this section we drop the assumption of a common distribution for all the component lifelengths. We assume the distribution to be the same within a cut set, but possibly different among cut sets. In many situations this may be a more realistic assumption; however it makes the model mathematically more complex. Throughout this section let X_{ij}, $j = 1, \ldots, n_i$, be the lifelengths of the components in cut set C_i with common continuous distribution F_i, $i = 0, 1, \ldots, k$. Let all the X_{ij}'s be independent. Finally, let $P^r(n_0, F_0; \underline{n}, \underline{F})$ denote the probability that cut set C_0 fails $r^{\underline{th}}$, $r = 1, \ldots, k+1$, where $\underline{F} = (F_1, \ldots, F_n)$. We derive an expression for $P^r(n_0, F_0, \underline{n}, \underline{F})$ in the following theorem, but we only study in detail the special case $r = 1$.

Theorem 4.1. Let $1 \le r \le k+1$. Then

(4.1) $P^r(n_0, F_0; \underline{n}, \underline{F}) =$

$$\sum_{S:|S|=r-1} \int_0^\infty \left[\prod_{i\in S}(F_i(x))^{n_i}\right]\left[\prod_{j\in S'}[1-(F_j(x))^{n_j}]\right] d(F_0(x))^{n_0}.$$

Proof. Again, as in the proof of Theorem 3.3, let $X_i^* = \max_{1 \le j \le n_i} X_{ij}$, $i = 0, \ldots, k$. Then $P^r(n_0, F_0; \underline{n}, F) = \sum_{S:|S|=r-1} P(\max_{i \in S} X_i^* < X_0^* < \min_{j \in S'} X_j^*)$. The independence of the X_i^*'s and a standard conditioning argument yields (4.1). ||

In particular for r = 1, (4.1) takes the form

$$P^1(n_0, F_0, \underline{n}, \underline{F}) = \int_0^\infty \prod_{i=1}^k [1 - (F_i(x))^{n_i}] d(F_0(x))^{n_0}. \tag{4.2}$$

We now utilize (4.2) to express certain intuitively obvious properties of $P^1(n_0, F_0; \underline{n}, \underline{F})$ in a precise form. As before, such properties are useful as a theoretical basis for setting up maintenance policies for this more realistic model.

We first note that for fixed n_0, F_0, $P^1(n_0, F_0; \underline{n}, \underline{F})$ remains unchanged if the same permutation is applied to $\underline{n}$ and $\underline{F}$. Next it is intuitively obvious that $P^1(n_0, F_0; \underline{n}, \underline{F})$ must posses certain monotonicity properties in its arguments. Before we express these properties in the next theorem, we state

Definition 4.2. A distribution function F is said to precede a distribution function G (in symbols $F \le G$) if $F(x) \le G(x)$ for all $x \in R$.

Notice that the above definition gives a partial order on the space $\mathcal{F}$ of all distribution functions, which is often referred to as F (or the random variable having distribution F) stochastically larger than G (or the random variable having distribution G).

We are now ready to state and prove Theorem 4.3 (a).

(a) For each F_0, $\underline{n}$, $\underline{F}$, $P^1(n_0, F_0; \underline{n}, \underline{F})$ is decreasing in n_0.

(b) For each n_0, F_0, $\underline{F}$, $P^1(n_0, F_0; \underline{n}, \underline{F})$ is increasing in each of the arguments $n_1, \ldots, n_k$.

(c) For each n_0, $\underline{n}$, $\underline{F}$, $P^1(n_0, F_0; \underline{n}, \underline{F})$ is increasing in F_0.

(d) For each n_0, F_0, $\underline{n}$, $P^1(n_0, F_0; \underline{n}, \underline{F})$ is decreasing in each of the arguments $F_1, \ldots, F_n$, where monotonicity in $\underline{F}$ is taken with respect to the partial ordering $\leq$.

<u>Proof</u>. The proofs of parts (b) and (d) are immediate from equation (4.2). To prove (a) and (c) we use an alternative expression for (4.2), namely

$$(4.3) \qquad P^1(n_0, F_0; \underline{n}, \underline{F}) = \int_0^\infty (F_0(x))^{n_0} d[1 - \prod_{i=1}^{k} (1 - (F_i(x))^{n_i})].$$

Again the proofs of (a) and (c) are now immediate from (4.3).

To state and prove the remaining theorem in this section which shows that $P^1(n_0, F_0, \underline{n}, \underline{F})$ has a further special ordering property, we recall the concept of a function decreasing in transposition. First we give the following definition.

<u>Definition 4.3</u>. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$, and $\underline{x} = (x_1, \ldots, x_k)$, be two vectors in R_k, the k-dimensional Euclidean space. A function $g\colon R_k \times R_k \to R$ is said to be <u>decreasing in transposition</u> (DT) if (i) $g(\underline{\lambda}, \underline{x})$ is unchanged when the same permutation is applied to $\underline{\lambda}$ and to $\underline{x}$, and (ii) $g(\underline{\lambda}, \underline{x}) \geq g(\underline{\lambda}, \underline{x}')$ whenever $\underline{x}'$ and $\underline{x}$ differ in two coordinates only, say i and j, $(i - j)(x_i - x_j) \geq 0$, and $x_i' = x_j$, $x_j' = x_i$. (See Hollander, Proschan and Sethuraman (1977) for a study of DT functions and their applications to ranking problems.)

We now extend the above definition.

<u>Definition 4.4</u>. Let X be a partially ordered set and X_k be its Cartesian product of order k. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$, be a vector in $M_k \subseteq R_k$. Let $G\colon M_R \times X_R \to R$ such that (i) $g(\underline{\lambda}, \underline{x})$ is unchanged when the same permutation is applied to $\underline{\lambda}$ and to $\underline{x}$, and

(ii) $g(\underline{\lambda}, \underline{x}) \geq g(\underline{\lambda}, \underline{x}')$ whenever $\underline{x}'$ and $\underline{x}$ differ in two coordinates only, say i and j, $i \leq j$, $x_i \leq x_j$, $x_i' = x_j$, and $x_j' = x_i$. The function g is then said to be <u>decreasing in transposition</u>.

Before we state and prove Theorem 4.7 we give the following definition and lemma.

<u>Definition 4.5</u>. A function $\phi(\lambda, x)$ on R_2 is <u>totally positive of order 2</u> (TP_2) if (a) $\phi(\lambda, x) \geq 0$, and (b) $\lambda_1 < \lambda_2$, $x_1 < x_2$ imply that $\phi(\lambda_1, x_1)\phi(\lambda_2, x_2) \geq \phi(\lambda_1, x_2)\phi(\lambda_2, x_1)$.

For instance, see Karlin (1968), Chap. 1. It can be easily verified that $\phi(x, y)$ is TP_2 if and only if $\frac{\partial^2}{\partial y \partial x} (\log \phi(x, y)) \geq 0$ (provided the partial derivative exists). Also note that the domain of the definition of ϕ need not be all of R_2.

The following lemma establishes the fact that a certain function is TP_2, this fact is needed in the proof of Theorem 4.7.

<u>Lemma 4.6</u>. Let $f(x, y) = 1 - x^y$, $0 \leq x \leq 1$, $y > 0$. Then $f(x, y)$ is TP_2.

<u>Proof</u>. We prove the lemma by showing that $\frac{\partial^2}{\partial y \partial x} [\log(1 - x^y)] \geq 0$. This in turn is easily established by simple algebra and using the well-known inequality $a - 1 - \log a \geq 0 \quad 0 \leq a \leq 1$.

We are now ready to state and prove the following theorem.

<u>Theorem 4.7</u>. For each n_0, F_0, $P^1(n_0, F_0; \underline{n}, \underline{F})$ is decreasing in transposition.

<u>Proof</u>. As noted previously, $P^1(n_0, F_0; \underline{n}, \underline{F})$ remains unchanged when the same permutation is applied to $\underline{n}$ and $\underline{F}$. Now let $n_1 \leq n_2 \leq \ldots \leq n_k$ and let $\underline{F}$, $\underline{F}'$ be two vectors of distribution functions ($\underline{F}, \underline{F}' \in F_k$) which differ only in

two coordinates i and j say, $i < j$, $F_i \le F_j$, $F_i' = F_j$, and $F_j' = F_i$. By Equation (4.2) we have,

$$P'(n_0, F_0, \underline{n}, \underline{F}) - P^1(n_0, F_0, \underline{n}, \underline{F}') =$$

$$\int_0^\infty [\prod_{\substack{t=1 \\ t \notin \{i,j\}}}^{k} (1 - (F_t(x))^{n_t})][(1 - (F_i(x))^{n_i}(1 - (F_j(x)^{n_j})$$

$$- (1 - (F_j(x))^{n_i})(1 - F_i(x))^{n_j}]d(F_0(x))^{n_0}.$$

The proof of the theorem will be completed when we show that $h(x) = [1 - (F_i(x))^{n_i}][1 - (F_j(x))^{n_j}] - [1 - (F_i(x)^{n_j}][1 - (F_j(x))^{n_i}] \ge 0$. But this follows from Lemma 4.6 and the fact that $n_i \le n_j$ and $F_i(x) \le F_j(x)$ for all x.

The above theorem expresses in a precise form the following intuitively obvious idea: It is easier for C_0 to fail first when the smaller size cut sets have components with stochastically larger lifelengths.

BIBLIOGRAPHY

El-Neweihi, E., Proschan, F. and Sethuraman, J. (1978). A simple model with applications in structural reliability, extinction of species, inventory depletion and urn sampling. Adv. Appl. Prob. 10, 232-254.

Hardy, G. H., Littlewood, J. E. and Pólya, G. (1952). Inequalities. Cambridge University Press.

Hollander, M., Proschan, F. and Sethuraman, J. (1977). Functions decreasing in transposition and their applications in ranking problems. Ann. Statist. 5, 722-733.

Karlin, S. (1968). Total Positivity, Vol. I. Stanford University Press.